Binary representations of multi-state monotone systems

Edward Korczak
Telecommunications Research Institute
ul. Poligonowa 30, 04-051 Warsaw, Poland
korczak@pit.edu.pl

Abstract

A new definition of a binary representation of a multistate monotone structure function is formulated, and its main properties are presented. Several forms of the binary representation and its factoring are discussed. Applicability of the results is illustrated by an example.

1. Introduction

Although the theory of binary systems has many practical applications, it is being replaced by the theory of multistate systems (MSS). In fact, many modern systems (and their elements as well) are capable of assuming a whole range of performance levels, varying from perfect functioning to complete failure. The present state-of-art of the theory and practice of MSS may be found in Lisnianski and Levitin (2003). Reliability analysis of MSS is difficult, therefore some effort has been made in applying existing binary methods to MSS. In particular, Block and Savits (1982) introduced two binary representations of MSS, decomposing a multistate structure into a sequence of binary structures. Similar decompositions were considered in Butler (1982) and Natvig (1982).

In this paper a new definition of a binary representation of an MSS is formulated and its main properties are obtained. The definition is given in terms of conditions to be satisfied, whereas the definition proposed by Block and Savits (1982) is given by a closed-form formula. Several forms of the binary representation are discussed. Problem of variables' relevance is also addressed. Furthermore, two different types of factoring of binary representations are introduced, which can be used to simplify the corresponding binary structures. Finally, the application of the binary representation to the calculation of the reliability indices is illustrated by an example.

2. Basic definitions

Let $<\boldsymbol{C}, \mathbf{K}, \mathbf{K}_1, ..., \mathbf{K}_n, \varphi>$ be a multistate system consisting of n multistate elements with the index set $\boldsymbol{C} = \{1, 2, ..., n\}$, where $\mathbf{K} = \{0, 1, ..., M\}$ is the set of the system states, $\mathbf{K}_i = \{0, 1, ..., M_i\}$ is the set of the states of element $i \in \boldsymbol{C}$, and φ: $\mathbf{V} \to \mathbf{K}$ is the system structure function, where $\mathbf{V} = \mathbf{K}_1 \times \mathbf{K}_2 \times ... \times \mathbf{K}_n$ is the space of element state vectors. The state of the system is determined by the element state vector and the structure function: $\varphi(\mathbf{x})$ is the system state when the element state vector is $\mathbf{x}$.

We assume that the states of the system [element i] represent successive levels of performance ranging from the perfect functioning level M [M_i] down to the complete failure level 0, i.e. the state spaces of the system and its elements are totally ordered. The system is a *multistate monotone system* (MMS) if its structure function φ is non-decreasing in each argument, $\varphi(\mathbf{0})=0$ and $\varphi(\mathbf{M})=M$, where $\mathbf{0} = (0,...,0)$, $\mathbf{M} = (M_1,...,M_n)$. A vector $\mathbf{y} = (y_1,...,y_n) \in \mathbf{V}$ is said to be a *path vector to level j* of an MMS iff $\varphi(\mathbf{y}) \geq j$. It is called a *minimal path vector to level j* if in addition $\mathbf{x} < \mathbf{y}$ implies $\varphi(\mathbf{x}) < j$. where $\mathbf{x} < \mathbf{y}$ means $x_i \leq y_i$ for i=1,...,n, and $x_i < y_i$ for some i. The set of all minimal path vectors to level j is denoted by

$\mathbf{U}_j$, where $\mathbf{U}_0 = \{\mathbf{0}\}$. A vector $\mathbf{z} = (y_1,...,y_n) \in \mathbf{V}$ is said to be a *cut vector to level j* of an MMS iff $\varphi(\mathbf{y}) < j$. It is called a *minimal cut vector to level j* if in addition $\mathbf{z} < \mathbf{x}$ implies $\varphi(\mathbf{x}) \geq j$. The set of all minimal cut vectors to level j is denoted by $\mathbf{L}_j$, where $\mathbf{L}_0 = \varnothing$. The dual structure φ^D is given by $\varphi^D(\mathbf{x}) = M - \varphi(\mathbf{M}-\mathbf{x})$.

3. Binary representation of an MMS and its properties

Let $J_{i;r}(x_i) = I(x_i \geq r)$, $x_i \in \mathbf{K}_i$, $i \in \boldsymbol{C}$, $r \in \mathbf{K}_i - \{0\}$, where $I(\cdot)$ is the indicator function. Define vector-valued functions $\mathbf{J}_i : \mathbf{K}_i \to \{0,1\}^{M_i} = \mathbf{F}_i$ $(i \in \boldsymbol{C})$ and $\mathbf{J} : \mathbf{V} \to \{0,1\}^{M_1} \times ... \times \{0,1\}^{M_n} = \mathbf{F} = \mathbf{F}_1 \times ... \times \mathbf{F}_n$:

$$\mathbf{J}_i(x_i) = (J_{i;1}(x_i), ..., J_{i;M_i}(x_i)),\ \mathbf{J}(\mathbf{x}) = (\mathbf{J}_1(x_1), ..., \mathbf{J}_n(x_n)) = (J_{i;r}(x_i): i \in \boldsymbol{C}, r \in \mathbf{K}_i - \{0\}). \tag{1}$$

Elements of the set $\mathbf{F}_i$ are denoted as vectors with single underlying: $\underline{\mathbf{x}}_i = (x_{i;1}, x_{i;2}, ..., x_{i;M_i})$, where $x_{i;r} \in \{0, 1\}$. Elements of the set $\mathbf{F}$ are denoted as vectors with double underlying: $\underline{\underline{\mathbf{x}}} = (\underline{\mathbf{x}}_1, ..., \underline{\mathbf{x}}_n) = \{x_{i;r}\}$. Let $\boldsymbol{\Delta}_i = \mathbf{J}_i(\mathbf{K}_i) \subseteq \mathbf{F}_i$ and $\boldsymbol{\Delta} = \mathbf{J}(\mathbf{V}) = \boldsymbol{\Delta}_1 \times ... \times \boldsymbol{\Delta}_n \subseteq \mathbf{F}$. We have:

$$\boldsymbol{\Delta}_i = \{\underline{\mathbf{x}}_i = (x_{i;1}, x_{i;2}, ..., x_{i;M_i}) : 1 \geq x_{i;1} \geq x_{i;2} \geq ... \geq x_{i;M_i} \geq 0\} = \{\underline{\mathbf{e}}_i(r) : r \in \mathbf{K}_i\}, \tag{2}$$

where $\underline{\mathbf{e}}_i(r) = (I(a \leq r): a = 1, ..., M_i) = \mathbf{J}_i(r)$. Hence $\mathbf{J}(\mathbf{x}) = (\underline{\mathbf{e}}_i(x_i) : i \in \boldsymbol{C})$.

Definition. Let φ_j: $\mathbf{F} \to \mathrm{R}$, $j \in \mathbf{K} - \{0\}$, be real functions of binary variables $\underline{\underline{\mathbf{x}}} = \{x_{i;r}\}$. We say that $\{\varphi_j\}$ is a *binary representation* of an MMS φ if $\forall \mathbf{x} \in \mathbf{V}$ $\forall j \in \mathbf{K} - \{0\}$: $I(\varphi(\mathbf{x}) \geq j) = \varphi_j(\mathbf{J}(\mathbf{x}))$.

From the definition and properties of φ and $\mathbf{J}$ it follows that:

(a) $\{\varphi_j\}$ are non-trivial binary monotone functions on $\boldsymbol{\Delta}$.

(b) $1 \geq \varphi_1 \geq \varphi_2 \geq ... \geq \varphi_M \geq 0$ on $\boldsymbol{\Delta}$.

(c) $\forall \mathbf{x} \in \mathbf{V}$ $\varphi(\mathbf{x}) = \sum_{j=1}^{M} \varphi_j(\mathbf{J}(\mathbf{x}))$.

(d) A binary representation is determined uniquely on $\boldsymbol{\Delta}$, i.e. if $\{\varphi_j\}$ and $\{\psi_j\}$ are two binary representations of an MMS, then $\varphi_j = \psi_j$ on $\boldsymbol{\Delta}$ for each $j \in \mathbf{K} - \{0\}$.

(e) Let $\{\phi_j\}$, j=1,...,M, be pseudo-boolean functions defined on $\mathbf{F}$. If $\{\phi_j\}$ satisfy (a) and (b), then $\phi(\mathbf{x}) = \sum_{j=1,...,M} \phi_j(\mathbf{J}(\mathbf{x}))$ is an MMS and $\{\phi_j\}$ is its binary representation.

(f) A binary representation $\{\varphi_j^D\}$ of the dual structure φ^D can be obtained from a binary representation $\{\varphi_j\}$ of φ as follows: (1) obtain the pseudo-dual functions $\varphi_j^{d1}(\{x_{i;r}\}) = 1 - \varphi_j(\{1 - x_{i;r}\})$; (2) obtain functions $\{\varphi_j^{d2}\}$, by replacing each $x_{i;r}$ with $x_{i;M_i - r+1}$ in algebraic expressions defining functions $\{\varphi_j^{d1}\}$; (3) set $\varphi_j^D = \varphi_{M-j+1}^{d2}$, $j = 1, ..., M$.

The function φ_j may be written in several forms equivalent on $\boldsymbol{\Delta}$, but not necessarily on $\mathbf{F}-\boldsymbol{\Delta}$. Some of these forms are direct generalizations of the forms known from the binary system theory. The algebraic expression defining the function φ_j may be treated as in the binary system theory, using techniques and tools known from this theory. Moreover, due to the restrictions on $x_{i;r}$ imposed by (2), we may use several additional simplification rules which do not alter the function φ_j on the set $\boldsymbol{\Delta}$. The most important rules are:

$$x_{i;r} x_{i;s} = x_{i;r} \wedge x_{i;s} = x_{i;\max(r,s)},\quad x_{i;r} \vee x_{i;s} = x_{i;\min(r,s)},\quad \overline{x}_{i;r} x_{i;s} = x_{i;s} - x_{i;\min(r,s)}, \tag{3}$$

where, as usual, $a \wedge b = \min(a, b) = ab$, $a \vee b = \max(a, b) = a+b-ab$ and $\bar{a} = 1 - a$) for any binary a and b.

Block and Savits (1982) defined binary representations using so-called, *min-path* and *min-cut forms*:

$$\varphi_j^{(\mathrm{MP})}(\underline{\underline{\mathbf{x}}}) = \max_{\mathbf{y}\in\mathbf{U}_j} \min_{i\in\boldsymbol{C}:\, y_i>0} x_{i;y_i}, \quad \varphi_j^{(\mathrm{MC})}(\underline{\underline{\mathbf{x}}}) = \min_{\mathbf{z}\in\mathbf{L}_j} \max_{i\in\boldsymbol{C}:\, z_i<M_i} x_{i;z_i+1}. \tag{4}$$

By applying the inclusion-exclusion principle to (4) and then the first rule of (3) to each term, we obtain the so-called *Poincaré's form*:

$$\varphi_j^{(\mathrm{P})}(\underline{\underline{\mathbf{x}}}) = \sum_{\varnothing\neq\mathbf{D}\subseteq\mathbf{U}_j} (-1)^{|\mathbf{D}|+1} \prod_{i\in\boldsymbol{C}:\, u_i(\mathbf{D})>0} x_{i;u_i(\mathbf{D})} = \sum_{k=1}^{|\mathbf{U}_j|}(-1)^{k+1} \sum_{\mathbf{D}\subseteq\mathbf{U}_j:\,|\mathbf{D}|=k} \prod_{i\in\boldsymbol{C}:\, u_i(\mathbf{D})>0} x_{i;u_i(\mathbf{D})}, \tag{5}$$

where for any $\varnothing \neq \mathbf{D} \subseteq \mathbf{U}_j$, $u_i(\mathbf{D}) = \max\{x_i\text{: } \mathbf{x}\in\mathbf{D}\}$, $i\in\boldsymbol{C}$, $|\mathbf{D}| = \mathrm{card}(\mathbf{D})$.

Among many other forms, there is a *pseudo-polynomial form*:

$$\varphi_j^{(\mathrm{PP})}(\underline{\underline{\mathbf{x}}}) = \alpha_0 + \sum_{k=1}^{m} \alpha_k B_k(\underline{\underline{\mathbf{x}}}), \quad B_k(\underline{\underline{\mathbf{x}}}) = \prod_{i\in\boldsymbol{C}} x_{i;a_{k,i}} \bar{x}_{i;b_{k,i}} =_{\boldsymbol{\Delta}} \prod_{i\in\boldsymbol{C}} (x_{i;a_{k,i}} - x_{i;b_{k,i}}), \tag{6}$$

where α_k are integer coefficients $0 \le a_{k,i} < b_{k,i} \le M_i+1$ for all i and k, $=_{\boldsymbol{\Delta}}$ means equality on $\boldsymbol{\Delta}$, $x_{i,0} \equiv 1$, $\bar{x}_{i,M_i+1} = 1 - x_{i,M_i+1} \equiv 1$, and the products B_k are non-trivial. Observe that $\mathbf{J}^{-1}(B_k^{-1}(1)) = [\mathbf{a}_k, \mathbf{b}_k - \mathbf{1}]$ is a multidimensional interval in $\mathbf{V}$, where $\mathbf{a}_k = (a_{k,1},...,a_{k,n})$ and $\mathbf{b}_k = (b_{k,1},...,b_{k,n})$. Two product $B_k(\underline{\underline{\mathbf{x}}})$ and $B_l(\underline{\underline{\mathbf{x}}})$ are orthogonal, if $B_k(\underline{\underline{\mathbf{x}}})B_l(\underline{\underline{\mathbf{x}}}) = 0$ on $\boldsymbol{\Delta}$, i.e. when the corresponding intervals are disjoint. If in (6), any two products are disjoint, $\alpha_0 = 0$ and $\alpha_k = 1$ (k=1,...,m), then (6) called the *orthogonal form*, or *SDP form*. A particular case of the orthogonal form is the *canonical disjunctive normal form*, in which the products correspond to single-point intervals.

We say that variable $x_{i;r}$ is relevant for the function φ_j if there exists $\underline{\underline{\mathbf{x}}} \in \boldsymbol{\Delta}$ such that $(0_{(i;r)}, \underline{\underline{\mathbf{x}}}) \in \boldsymbol{\Delta}$, $(1_{(i;r)}, \underline{\underline{\mathbf{x}}}) \in \boldsymbol{\Delta}$ and $\varphi_j(0_{(i;r)}, \underline{\underline{\mathbf{x}}}) \neq \varphi_j(1_{(i;r)}, \underline{\underline{\mathbf{x}}})$, or equivalently, if there exists $\underline{\underline{\mathbf{x}}} \in \boldsymbol{\Delta}$ such that $\varphi_j(\underline{\mathbf{e}}_i(r), \underline{\underline{\mathbf{x}}}) \neq \varphi_j(\underline{\mathbf{e}}_i(r-1), \underline{\underline{\mathbf{x}}})$, where $(\underline{\mathbf{e}}_i(r), \underline{\underline{\mathbf{x}}}) = (\underline{\mathbf{x}}_1, ..., \underline{\mathbf{e}}_i(r), ..., \underline{\mathbf{x}}_n)$. The functions $\{\varphi_j\}$ may have many irrelevant variables. However in many practical cases, the algebraic expressions defining these functions contain relevant variables only. For example, all variables in the expressions shown (4) and (5) are relevant. Moreover, it can be shown that the set of relevant variables for φ_j is:

$$\{\text{relevant variables of } \varphi_j\} = \{x_{i;y_i} : i \in \boldsymbol{C}, y_i > 0, \mathbf{y} \in \mathbf{U}_j\} = \{x_{i;z_i+1} : i \in \boldsymbol{C}, z_i < M_i, \mathbf{z} \in \mathbf{L}_j\}. \tag{7}$$

The factoring is a popular technique for analyzing complex binary systems. There are two kinds of factoring (pivotal decompositions) of the function φ_j: with respect to the entire vector $\underline{\mathbf{x}}_i \in \boldsymbol{\Delta}_i$ (or the state of element i), and with respect to a single variable $x_{i;r}$. Considering all possible values of vector $\underline{\mathbf{x}}_i \in \boldsymbol{\Delta}_i$, we obtain the factoring formula with respect to the state of element i:

$$\varphi_j(\underline{\underline{\mathbf{x}}}) = \sum_{r\in\mathbf{K}_i} I(\underline{\mathbf{x}}_i = \underline{\mathbf{e}}_i(r))\varphi_j(\underline{\mathbf{e}}_i(r), \underline{\underline{\mathbf{x}}}) = \sum_{r\in\mathbf{K}_i} x_{i;r}\bar{x}_{i;r+1}\varphi_j(\underline{\mathbf{e}}_i(r), \underline{\underline{\mathbf{x}}}) = \sum_{r\in\mathbf{K}_i}(x_{i;r} - x_{i;r+1})\varphi_j(\underline{\mathbf{e}}_i(r), \underline{\underline{\mathbf{x}}}) \tag{8}$$

for $\underline{\underline{\mathbf{x}}} \in \boldsymbol{\Delta}$.

The factoring with respect to a single variable $x_{i;r}$ is given by:

$$\begin{aligned} \varphi_j(\underline{\underline{\mathbf{x}}}) &= x_{i;r}\,\varphi_j(1_{(i;r)}, \underline{\underline{\mathbf{x}}}) + \bar{x}_{i;r}\,\varphi_j(0_{(i;r)}, \underline{\underline{\mathbf{x}}}) \\ =_{\boldsymbol{\Delta}}\ \varphi_j(\underline{\underline{\mathbf{x}}}) &= x_{i;r}\,\varphi_j((1_{(i;s)} : 1 \le s \le r), \underline{\underline{\mathbf{x}}}) + \bar{x}_{i;r}\,\varphi_j((0_{(i;s)} : r \le s \le M_i), \underline{\underline{\mathbf{x}}}). \end{aligned} \tag{9}$$

The second equality follows from the fact that on $\boldsymbol{\Delta}$, the condition $x_{i;r} = 1$ means $x_{i;s} = 1$ for all $1 \le s \le r$, and the condition $x_{i;r} = 0$ means $x_{i;s} = 0$ for all $r \le s \le M_i$. It can be seen that recursive application of (9) with respect to $x_{i;M_i}$, ..., $x_{i;1}$ successively, leads to (8).

4. Example of application

Suppose that the states of elements 1, ..., n of an MMS are represented by s-independent random variables X_1, ..., X_n. Then the state of the MMS is $\varphi(\mathbf{X})$, where $\mathbf{X} = (X_1, ..., X_n)$. It is clear that for any fixed $i \in \boldsymbol{C}$ the random variables $X_{i;r} = J_{i;r}(X_i)$, $r \in \mathbf{K}_i - \{0\}$ are s-dependent, as $X_{i;r} \geq X_{i;s}$ for $r<s$. However, the random vectors $(X_{1;r} : r \in \mathbf{K}_1 - \{0\})$, ..., $(X_{n;r} : r \in \mathbf{K}_n - \{0\})$ are s-independent. Let $R(j) = \Pr\{\varphi(\mathbf{X}) \geq j\} = E[\varphi_j(\mathbf{J}(\mathbf{X}))]$ be the system reliability to the level j and $R_i(r) = \Pr\{X_i \geq r\} = E[X_{i;r}]$ be the reliability to the level r of element i. Assuming that $\{R_i(r)\}$ are known, and that the function φ_j is given in the form (6), the calculation of $R(j)$ is very easy:

$$R(j) = E[\varphi_j^{(\mathrm{PP})}(\mathbf{J}(\mathbf{X}))] = \alpha_0 + \sum_{k=1}^{m} \alpha_k E[B_k(\mathbf{J}(\mathbf{X}))], \quad E[B_k(\mathbf{J}(\mathbf{X}))] = \prod_{i \in \boldsymbol{C}} (R_i(a_{k,i}) - R_i(b_{k,i})), \tag{10}$$

where clearly $R_i(0) \equiv 1$ and $R_i(M_i+1) \equiv 0$. It is seen that the choice of a suitable binary representation of an MMS overcomes difficulties with dependence of random variables belonging to the same element.

Example. Consider an MMS with $\mathbf{K} = \{0, 1, 2, 3\}$, $\mathbf{K}_1 = \mathbf{K}_2 = \{0, 1, 2\}$, $\mathbf{K}_3 = \{0, 1, 2, 3\}$. The structure φ is defined by the sets of minimal path vectors: $\mathbf{U}_1 = \{(1,2,1), (2,1,1), (0,1,2), (1,0,2), (0,0,3)\}$, $\mathbf{U}_2 = \{(1,1,2), (0,1,3), (1,0,3)\}$ and $\mathbf{U}_3 = \{(1,2,3), (2,1,3)\}$. We demonstrate the use of the factoring of φ_1 with respect to element 3. According (4):

$$\varphi_1(\underline{\underline{\mathbf{x}}}) = x_{1;1}x_{2;2}x_{3;1} \vee x_{1;2}x_{2;1}x_{3;1} \vee x_{2;1}x_{3;2} \vee x_{1;1}x_{3;2} \vee x_{3;3} . \tag{11}$$

We have:

$$\varphi_1(\underline{\mathbf{e}}_3(0), \underline{\underline{\mathbf{x}}}) = x_{1;1}x_{2;2}0 \vee x_{1;2}x_{2;1}0 \vee x_{2;1}0 \vee x_{1;1}0 \vee 0 = 0, \tag{12}$$

$$\varphi_1(\underline{\mathbf{e}}_3(1), \underline{\underline{\mathbf{x}}}) = x_{1;1}x_{2;2}1 \vee x_{1;2}x_{2;1}1 \vee x_{2;1}0 \vee x_{1;1}0 \vee 0 = x_{1;1}x_{2;2} + x_{1;2}(x_{2;1} - x_{2;2}), \tag{13}$$

$$\varphi_1(\underline{\mathbf{e}}_3(2), \underline{\underline{\mathbf{x}}}) = x_{1;1}x_{2;2}1 \vee x_{1;2}x_{2;1}1 \vee x_{2;1}1 \vee x_{1;1}1 \vee 0 = x_{1;1} + (1 - x_{1;1})x_{2;1}, \tag{14}$$

$$\varphi_1(\underline{\mathbf{e}}_3(3), \underline{\underline{\mathbf{x}}}) = x_{1;1}x_{2;2}1 \vee x_{1;2}x_{2;1}1 \vee x_{2;1}1 \vee x_{1;1}1 \vee 1 = 1. \tag{15}$$

Thus, by (8)

$$\varphi_1(\underline{\underline{\mathbf{x}}}) = (x_{3;1} - x_{3;2})[x_{1;1}x_{2;2} + x_{1;2}(x_{2;1} - x_{2;2})] + (x_{3;2} - x_{3;3})[x_{1;1} + (1 - x_{1;1})x_{2;1}] + x_{3;3} . \tag{16}$$

Finally,

$$\begin{aligned} R(1) = E[\varphi_1(\mathbf{J}(\mathbf{X}))] &= (R_3(1) - R_3(2))[R_1(1)R_2(2) + R_1(2)(R_2(1) - R_2(2))] \\ &+ (R_3(2) - R_3(3))[R_1(1) + (1 - R_1(1))R_2(1)] + R_3(3) . \end{aligned} \tag{17}$$

References

Butler, D.A. (1982). Bounding the reliability of multistate systems. *Operations Research 30* (3), 530–544.

Block, H.W. and Savits, T.H. (1982). A decomposition for multistate monotone systems. *Journal of Applied Probability 19* (2), 391–402.

Lisnianski, A. and Levitin, G. (2003). Multi-state system reliability. Assessment, optimization and applications. New Jersey: World Scientific.

Natvig, B. (1982). Two suggestions of how to define a multistate coherent system. *Advances in Applied Probability 14* (2), 434–455.